THE OPTIMUM USE OF ANTIBALLISTIC MISSILES FOR POINT TARGET DEFENSE

by

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THESIS

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October 1969

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
October 1969

ABSTRACT

This thesis is concerned with developing an optimal launching schedule for ABM's deployed for defense of a number of point targets, i.e. an ICBM complex. Let the attack occur in N stages with an offensive strategy of saturation. A dynamic programming model is developed for formulating the problem and a linear programming model is used in its solution. Equations are developed for determining the optimal number of ABM's to launch on each stage of the attack so as to maximize the expected number of silos surviving. Multivalued point target defense is discussed and formulated but no specific solutions are offered. The ABM system (including radars, computers, etc.) is not considered subject to attack. Some discussion of this aspect of the problem is offered. Single ABM launchings per re-entry vehicle are assumed. A test is developed to determine for what parameters single launchings are preferred over multiple launchings, under the assumptions of the attack.

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I. INTRODUCTION

This thesis addresses the problem of efficient allocation of anti-ballistic missiles (ABM's) during an enemy attack on United States' land-based intercontinental ballistic missiles (ICBM's). The targets (ICBM silos) are treated as "point targets", i.e., the area of the target is small compared to the destructive capability of the attacking missiles. This paper is concerned with optimizing defensive strategy only, and reasonable assumptions are made concerning the strategy of the offense.

For a discussion of offensive and defensive strategy optimization during a single battle, the reader is referred to Perkins, F. M., Optimum Weapon Deployment for Nuclear Attack [Ref 1] and McEwen, W. R., The Attack and Defense of Targets by Missiles [Ref 2]. Perkins assumes total knowledge both by offensive and defensive forces, and a complex of point targets as the object of the battle, whereas McEwen discusses the problem of a single point target using both probabilistic and game theory models to obtain a solution. For a discussion of offensive strategy optimization only, refer to Piccariello, H. J., Missile Allocation [Ref 3] and McLaren, M. D., Walkup, P. W., A Missile Targeting Problem [Ref 4], and A Multiple Assignment Problem [Ref 5]. Piccariello takes into account the vulnerability of control centers as well as the silos. He develops solutions for the continuous and discrete cases. Both papers coauthored by McLaren and Walkup are closely related and should be read together. The first uses Monte Carlo techniques for estimating expected damage and minimum damage level for a given strategy. The second paper is concerned with optimal programming and targeting of missiles prior to an attack.

This paper considers the attack to occur in stages, the number of stages being dependent upon the size of the missile inventory of the offense. The question for the defense is, "What quantity of defensive forces (ABM's) should be expended at each stage of the attack in order to maximize the number (or value) of undestroyed silos after the attack?" Two general scenarios are considered. First, all silos (point targets) are considered to be equivalent in target value and second, all silos are not equivalent but rather belong to ordered classes. Both dynamic and linear programming methods were applied in an attempt to formulate and solve this problem. Although both of these methods were satisfactory in the formulation phase, they proved less efficient than a direct analytical approach in obtaining a closed form solution.

From a practical viewpoint, the solution should necessarily be restricted to being inter-valued, since it would be meaningless to require the firing of 1/2, 1/3, or any other fraction of an ABM at an attacking missile. This aspect of the problem is ignored but unlike Pennington's [Ref 6] solution to a similar problem, the integar condition is not thought to be critical in this analysis because of assumptions made concerning the geometry of the targets and the capabilities of the defense. Equations are developed, in terms of initial parameters, for the optimal allocation of the ABM's.

Interest was developed for this thesis by Pennington's paper [Ref 6] concerning the defense of a single point target against successive missile attacks. Unless a large number of ABM's were available for each point target, his solution often resulted in the firing of fractional ABM's. Although such a result is mathematically correct, it is not very useful. For example, firing 1/3 of an ABM at each of three attacking re-entry vehicles (RV's) results in a higher target survivability than firing one ABM at only one of those three attacking RV's (see Appendix A for a proof of this statement).

This analysis is restricted to single ABM launchings against any particular RV. If a sufficient number of ABM's is available to allow multiple launchings at a single RV, and to defend every undestroyed silo on every attack, then the equations developed herein are inapplicable. An exact relationship defining the region of applicability for this solution is developed in section (III.B.8). The general method of this paper could be extended to a situation involving large numbers of ABM's if a solution to that problem were desired. However, the practicality demanded by scarce resources indicates that this would probably not be the case. It seems unlikely that an attack would occur at all if the defense were so well endowed with antiballistic missiles. This implies a rational behavior on the part of the offense, which might or might not be justifiable in a real situation.

II. PROBLEM DESCRIPTION

A. ASSUMPTIONS

If and when an attack should ever occur against the United States' intercontinental ballistic missiles, and should an antiballistic missile system be installed to defend those ICBM's, there arises the obvious question, "How does the defense allocate ABM's so as to protect as many silos as possible from destruction?" In order to answer such a question it is necessary to make some assumptions about the nature of the attack, the reliability of the equipment, and the geometry of the situation.

A "farm" will be defined to be an area of land throughout which are scattered a number of silos containing ICBM's. The separation of the silos is sufficient to disallow multiple destruction by a single re-entry vehicle. If the re-entry vehicle is sufficiently close to one silo to destroy it, then it is too far from any other silo to cause its destruction. A re-entry vehicle is a single bomb and the fact that it might have previously been released from a warhead containing many bombs is not significant. Technology available to distinguish decoys and similar systems from actual re-entry vehicles, as well as the space and weight demanded by such systems preclude their consideration. Every radar target approaching the farm on a proper trajectory shall be treated as a re-entry vehicle.

The attack is to occur in stages, with the offense attacking every silo in the farm on every stage. The offense will not be allowed a "shoot-look-shoot" capability, i.e., the ability to ascertain battle damage between stages and adjust his strategy accordingly. Since each silo has the same probability of being undestroyed after the first stage, the offense is necessarily required to attack all the silos again on the second stage, or none. Unless the silos are of different strategic value (to be discussed in chapter IV) they are equivalent in the attacker's eyes on each stage of the attack. One further assumption justifies this reasoning more completely. Each ABM in the defensive system has the capability of defending any silo within the farm. Thus it is not possible for the offense to concentrate their attack so as to exhaust defenses in a portion of the farm. If there is any ABM left for defense, it can be launched to intercept an RV aimed at any silo within the farm. Thus the offense, if it attacks the farm at all, is assumed to attack each silo at each stage until its inventory is exhausted.

The silos, and the ICBM's they house, are the only targets attacked by the offense. All ICBM control centers, the ABM complex itself, and all associated radars are eliminated from consideration as targets. Chapter IV, part B, discusses this subject more completely.

It is assumed that the defense can ascertain battle damage between stages and thus defend only those silos which remain undestroyed. Once a silo is destroyed, it is left undefended on future stages of the attack, thus all RV's aimed at destroyed silos go unchallenged by the defense. If a RV successfully penetrates the defense but misses its preassigned target, it misses all targets. No lucky hits are allowed. In a similar fashion, it is assumed that one ABM can destroy only one RV. The spacing between offensive stages and between RV's within a stage is sufficient to disallow multiple successes by any ABM. The spacing between stages is not sufficient, however, to allow launchings of ICBM's during the attack. Only those ICBM's which survive the entire attack can be used by the defense in a retaliatory attack.

B. DEFINITIONS

Some terms as they are used in this paper have been defined in part A of this chapter. However, for ease of reference and completeness, all terms used which require an exact definition are listed below.

Point target: a target which has an overall surface area which is small when compared to the destructive capability of a single re-entry vehicle.

ABM: antiballistic missile.

ICBM: intercontinental ballistic missile.

RV: re-entry vehicle, a single fission or fusion bomb.

Stage: a single wave of RV's which all arrive at their targets at approximately the same time. There is one RV per target per wave.

100% defense: that defensive strategy such that all undestroyed silos are defended on a given stage.

Attack: a sequence of N independent stages, with stage N occurring first in time and stage one occurring at the end of the battle.

T_{N-i}: the expected number of undestroyed silos immediately prior to stage (N-i) of the attack.

X_{N-i}: the number of ABM's remaining in inventory immediately prior to stage (N-i) of the attack.

P_{N-i}: the probability that an ABM launched on stage (N-i) successfully intercepts and destroys an RV.

q: the probability that an undestroyed or unchallenged RV destroys its assigned target.

The fact that stage N of the attack is defined to occur first in time and stage one to occur at the end of the attack might seem strange at this point. The reason for this backward numbering comes from the dynamic programming formulation of the problem, and rather than use a different notation in other sections, it is used consistently for all formulations, and for the solution.

C. A DYNAMIC PROGRAMMING FORMULATION

1. Notation

At every stage of the attack, there is an expected number of silos destroyed. This destruction can occur for a number of reasons. First, every RV may not be challenged due to a scarcity of ABM's. Second, except as a limiting case, ABM reliability for successful interception is less than one. Of the RV's that go unchallenged and those that successfully penetrate the defenses, some will be successful in destroying their targets. Let Y_{N-i} (stage return) be defined as the expected number of silos destroyed on stage (N-i). A diagram of the entire attack is shown below in Fig. 2.1.

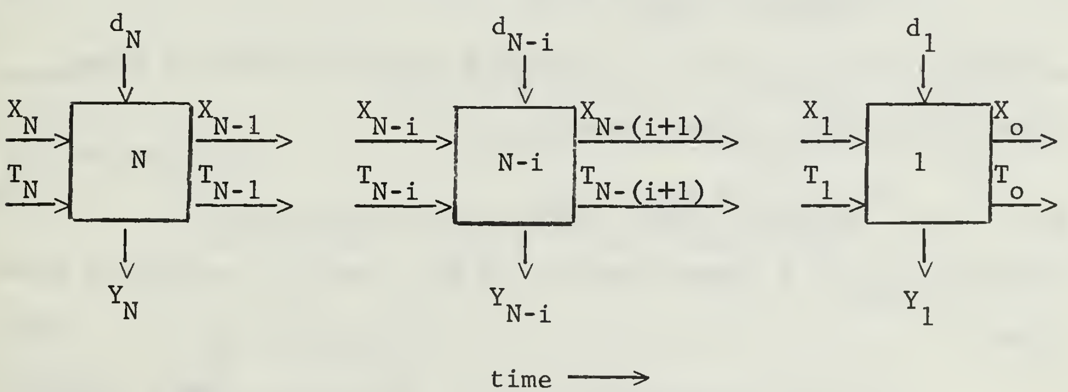

Figure 2.1

Each stage of the attack is identified by the index (N-i), i = 0, 1, ..., N-1. At the beginning of each stage, there are a number of ABM's still in inventory, X_{N-i}, and an expected number of silos still undestroyed, T_{N-i}. For this stage, a quantity of ABM's, d_{N-i}, is launched, leaving $X_{N-(i+1)}$ ABM's for future defense. Y_{N-i} silos are destroyed (expected number) leaving $T_{N-(i+1)}$ as the expected number of undestroyed silos for the beginning of the next stage. The d's are the decision variables which are to be selected in an "optimal" manner.

2. The Mathematical Model

The problem, as stated in these terms, is then to select the vector $\underline{d} = (d_1, d_2, \ldots, d_N)$ such that the sum of the expected number of silos lost on each stage is minimized, i.e.

$$\min \sum_{j=1}^{N} Y_j \quad \text{by selecting} \tag{2.1}$$

$d_1, d_2, \ldots, d_N$ in an optimal way.

a. The Stage Transformation Equations

Since d_{N-i} ABM's are launched on stage (N-i), the number remaining at stage (N-i)-1 is

$$X_{N-(i+1)} = X_{N-i} - d_{N-i} \tag{2.2}$$

and also since Y_{N-i} silos are expected to be destroyed on stage (N-i), the expected number remaining is

$$T_{N-(i+1)} = T_{N-i} - Y_{N-i}. \tag{2.3}$$

However, Y_{N-i} is a linear function of T_{N-i} and d_{N-i} as will be shown below.

(1) An Expression for Y_{N-i}. Of the d_{N-i} ABM's launched in defense of stage (N-i), p_{N-i} are expected to be successful in

destroying their assigned RV's. But T_N RV's are approaching the farm, of which T_{N-i} are directed at undestroyed silos. Since the defenders are concerned only with defense of undestroyed silos, the ABM's launched are directed at those T_{N-i} RV's aimed at the undestroyed silos, and the other RV's are left unchallenged. Thus, of the T_{N-i} RV's that are challenged $T_{N-i} - p_{N-i}\, d_{N-i}$ are expected to penetrate the defenses. Of these, the probability that each destroys its assigned silo is q and thus, the expected number of ICBM's destroyed on stage (N-i) is

$$Y_{N-i} = q(T_{N-i} - p_{N-i}\, d_{N-i}). \tag{2.4}$$

Therefore

$$T_{N-(i+1)} = T_{N-i} - q(T_{N-i} - p_{N-i}\, d_{N-i})$$

or

$$T_{N-(i+1)} = T_{N-i}(1 - q) + qp_{N-i}\, d_{N-i}. \tag{2.5}$$

3. <u>Analysis</u>

Let $$f_1(X_1, T_1) = \min_{d_1} (Y_1) \tag{2.6}$$

where $$X_1, T_1 \geq d_1 \geq 0$$

and $$f_2(X_2, T_2) = \min_{d_2} (Y_2 + f_1(X_1, T_1)) \tag{2.7}$$

where $$X_2, T_2 \geq d_2 \geq 0.$$

In general, let

$$f_j\,(X_j, T_j) = \min_{d_j} (Y_j + f_{j-1}\,(X_{j-1}, T_{j-1})) \tag{2.8}$$

where $X_j, T_j \geq d_j \geq 0.$

This expression, (2.8), along with the stage transformation equations, (2.2) and (2.5) is a dynamic programming formulation of the problem. Unfortunately, the problem cannot be solved in closed form as presented here. To realize this, consider eq. (2.6)

$$f_1\ (X_1, T_1) = \min_{d_1} (Y_1)$$

$$f_1\ (X_1, T_1) = \min_{d_1} (qT_1 - qp_1d_1).$$

Since the coefficient of d_1 is always non-positive, and negative for all practical cases, the quantity in parenthesis can be minimized by making d_1 as large as possible. However, d_1 cannot be larger than X_1 since no more ABM's can be launched than there are in inventory, and d_1 cannot be larger than T_1 since it is unnecessary to defend any silos which were previously destroyed. Now the matter becomes a question of whether $X_1 > T_1$ or $T_1 \geq X_1$.

Consider $X_1 > T_1$. In this case there would be more ABM's available for defense than were needed (recall only single ABM launchings are allowed against any given RV). This implies that the excess ABM's should have been used at an earlier stage as opposed to not using them at all. Any strategy, to be optimal, will use all available ABM's for defense. Thus X_1 must be equal to or less than T_1 and

$$d_1^* = X_1$$

where d_1^* is that value of d_1 which satisfies $f_1(X_1, T_1)$. Continuing with eq. (2.7)

$$f_2\ (X_2, T_2) = \min_{d_2} (Y_2 + f_1\ (X_1, T_1))$$

$$f_2\ (X_2, T_2) = \min_{d_2} (Y_2 + qT_1 - qp_1d_1^*) \tag{2.9}$$

where X_2, $T_2 \geq d_2 \geq 0$.

Equation (2.5) with i = N-2 is

$$T_1 = T_2 (1 - q) + qp_2d_2$$

and eq. (2.4) with i = N-2 is

$$Y_2 = q(T_2 - p_2d_2).$$

Then eq. (2.9) becomes

$$f_2 (X_2, T_2) = \min_{d_2} \left(q(2-q) T_2 - (qp_2 - q^2 p_2)d_2 - qp_1d_1^*\right).$$

Since $1 \geq q \geq 0$, the coefficient of d_2 is always non-positive, and d_2 must assume its maximum value in order to minimize the expression in parenthesis. This is where the problem arises if a closed form solution is desired. It is not known whether $X_2 > T_2$ or $T_2 \geq X_2$ and since the smaller of these two quantities is the upper bound for d_2, the maximum value of d_2 cannot be determined immediately. A tabular solution could be obtained once initial parameters were specified but a more direct solution is sought.

The following chapter gives two linear programming models of this problem. The second model is solved in closed form for all decision variables.

III. LINEAR PROGRAMMING FORMULATIONS

The problem addressed in this chapter is exactly the same as introduced earlier. The same assumptions regarding the structure of the attack and the defense will be used. All previously introduced notation is also unchanged, and additional notation is defined as required.

A. MODEL I

This model is presented for continuity and will not be solved in closed form. Therefore only necessary equations are shown and the details of their development are contained in Appendix B. However, the reader should be cautioned that there are many aspects of model I which are exactly the same as model II and an understanding of model I is necessary before going on to model II.

Let z represent the expected total number of silos lost after all N stages have occurred. Then

$$z = \sum_{j=1}^{N} Y_j \quad \text{(see (2.1))}. \tag{3.1}$$

However, each Y_j is a linear function of d_j and T_j as shown by eq. (2.4). It can further be shown (see Lemma 1) that each T_j is a linear function of d_i's, $i = j+1, \ldots, N$; thus, the expression for z above can be written

$$z = D_o + D_1 d_1 + \ldots + D_N d_N \quad \text{(see Appendix B)}. \tag{3.2}$$

Then the problem can be stated as follows

$$\min z = D_o + D_1 d_1 + \ldots + D_N d_N \tag{3.3}$$

subject to

$$(1)\ \sum_{i=0}^{N-1} d_{N-i} = X_N$$

$$(2)\ d_{N-i} \leq T_{N-i} \qquad i = 0, \ldots, N-1 \tag{3.4}$$

$$(3)\ d_{N-i} \geq 0 \qquad i = 0, \ldots, N-1.$$

The objective function, z, is linear. The coefficients of the variables (derived in Appendix B) are as follows

$$D_o = qT_N\left(\sum_{j=0}^{N-1} (1-q)^j\right)$$

$$D_1 = qp_1(-1)$$

$$\vdots$$

$$D_{N-i} = qp_{N-i}\left(q\sum_{j=0}^{N-(i+2)} (1-q)^j - 1\right) \tag{3.5}$$

$$\vdots$$

$$D_N = qp_N\left(q\sum_{j=0}^{N-2} (1-q)^j - 1\right).$$

1. Interpretation of Constraints

Constraint (1) requires the defense to launch all the ABM's in inventory and available. Constraint (2) is discussed in detail below and constraint (3) is a set of N non-negativity restrictions.

a. A Modification of Constraint (2)

Recall that the objective function is based upon the assumption that only one ABM is launched at any given RV. Constraint (2) assures that this assumption is not violated, but it must be modified to be useful since each T_j is a linear function of some of the decision variables, namely $d_{j+1}, \ldots, d_N$. Constraint (2) in the form used above is understandable in relation to the physical situation, but the form given below would have to be used if model I were to be solved using given parameters.

$$d_{N-i} - q \sum_{j=N-i+1}^{N} p_j d_j (1-q)^{j-(N-i+1)} \leq T_N (1-q)^i \qquad (3.6)$$

for $i = 0, 1, \ldots, N-1$.

This form of constraint (2) is stated as Lemma 2 in the next part of this chapter and a proof is given.

If one wished to do so, this model could be solved by means of a simplex algorithm. Naturally, all parameters must be specified. However, a general closed form solution for an equivalent model is presented in part B which offers a faster, more efficient solution.

B. MODEL II

1. Preliminaries

The two previous models used to formulate this allocation problem both utilized the concept of minimizing the expected losses or sum of expected losses for the N stages of the attack. Consider now an equivalent model which is developed in order to maximize the expected number of undestroyed silos at the end of the attack. Since the defenses' ICBM's are designed for retaliatory, or second strike capabilities, it is important to have as many as possible remaining to carry out that mission.

Recall that T_o is equal to the expected number of undestroyed silos after the farm has been subjected to an attack of N stages. The number of undestroyed silos prior to stage N is just T_N since none have yet been lost. It has been assumed that the offense attacks every silo on every stage, so the number of RV's aimed at undestroyed silos on stage N (first in time) is also equal to T_N.

2. Development of the Objective Function

Using eq. (2.5) with i=1 produces

$$T_{N-2} = T_{N-1}(1-q) + qp_{N-1}\, d_{N-1} \tag{3.7}$$

and for i=0 yields

$$T_{N-1} = T_N (1-q) + qp_N\, d_N. \tag{3.8}$$

Substituting eq. (3.8) into eq. (3.7) for T_{N-1} produces

$$T_{N-2} = T_N (1-q)^2 + q(1-q)\, p_N d_N + qp_{N-1} d_{N-1}. \tag{3.9}$$

Comparing eq. (3.9) with eq. (3.8) leads to the general formula of Lemma 1.

Lemma 1 If there are T_N silos to be defended at the beginning of an attack, then the expected number of silos remaining at the beginning of stage (N-i), i=1, ..., N-1; is

$$T_{N-i} = T_N(1-q)^i + q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j. \tag{3.10}$$

Proof Lemma 1 will be proved by induction. Note that eq. (3.8) is exactly the same as the equation produced by eq. (3.10) for i=1. Therefore Lemma 1 is valid for i=1. Assume Lemma 1 is valid for i=k. From chapter II, eq. (2.5) gives

$$T_{N-(k+1)} = T_{N-k}(1-q) + qp_{N-k} d_{N-k}. \tag{3.11}$$

Substituting eq. (3.10), using i=k, into eq. (3.11) for T_{N-k} yields

$$T_{N-(k+1)} = \left[T_N(1-q)^k + q \sum_{j=N-k+1}^{N} (1-q)^{j-(N-k+1)} p_j d_j\right](1-q) + qp_{N-k} d_{N-k}.$$

Combining terms produces

$$T_{N-(k+1)} = T_N(1-q)^{k+1} + q \sum_{j=N-k}^{N} (1-q)^{j-(N-k)} p_j d_j$$

which is eq. (3.10) for $i=k+1$, and which completes the proof for Lemma 1.

Therefore, the expression for T_o, which is the new objective function, is

$$\max T_o = T_N (1-q)^N + q \sum_{j=1}^{N} (1-q)^{j-1} p_j d_j . \tag{3.12}$$

3. Constraints

The constraints applicable here in model II are exactly the same as those given in model I, and will be restated following Lemma 2. Constraint (2) must be modified however, as mentioned in part A of this chapter.

Lemma 2 For each i; $i=0, 1, \ldots, N-1$; the following inequality is a restatement of constraint (2)

$$d_{N-i} - q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j \leq T_N (1-q)^i . \tag{3.13}$$

Proof Constraint (2) requires that, in general,

$$d_{N-i} \leq T_{N-i} \qquad i=0, 1, \ldots, N-1. \tag{3.14}$$

However, from Lemma 1

$$T_{N-i} = T_N (1-q)^i + q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j \qquad i=0, 1, \ldots, N-1.$$

Substituting this expression into eq. (3.14) yields

$$d_{N-i} \leq T_N (1-q)^i + q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j .$$

Rearranging terms

$$d_{N-i} - q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j \leq T_N (1-q)^i$$

which completes the proof of Lemma 2.

4. The Complete Form for Model II

In summary, model II is

$$\max T_o = T_N(1-q)^N + q \sum_{j=1}^{N} (1-q)^{j-1} p_j d_j$$

subject to

$$(1) \quad \sum_{i=0}^{N-1} d_{N-i} = X_N$$

$$(2) \quad d_{N-i} - q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j \leq T_N(1-q)^i \qquad (3.16)$$

$$i=0, \ldots, N-1$$

$$(3) \quad d_{N-i} \geq 0 \qquad i=0, \ldots, N-1.$$

It is this model from which the closed form solution is obtained.

5. The Solution

a. A Physical Consideration of the Objective Function

The objective function can be written more concisely as

$$T_o = T_N(1-q)^N + \sum_{j=1}^{N} C_j p_j d_j \qquad (3.17)$$

where $C_j = q(1-q)^{j-1}$. Note that $T_N(1-q)^N$ is a constant and has no effect on the optimization process once T_N and q are specified. Consider the case where

$$C_1 p_1 > C_2 p_2 > \ldots > C_N p_N. \qquad (3.18)$$

This inequality will be valid whenever

$$(1-q)^{j-1} p_j > (1-q)^j p_{j+1} \text{ for } j=1, \ldots, N-1$$

is true, or $p_j/p_{j+1} > (1-q)$.

The quantity p_j/p_{j+1} is a fraction between zero and one since it has previously been assumed that ABM reliability decreases in time. Also, practical values for q should be in the neighborhood of 0.6 to 1.0. Thus, if p deteriorated by no more than 60% per stage from its previous value, eq. (3.18) is valid. It is hoped that ABM reliability would not be reduced so drastically as to invalidate eq. (3.18).

b. A Characterization of the Optimal Solution

Theorem 1 If X_N ABM's are available for defense of an N-stage attack, and if $C_j p_j > C_i p_i$ for $1 \le j < i$; i=2, ..., N; then

$$d_i^* = 0 \quad \text{for } i=k+1, \ldots, N$$

$$d_k^* > 0$$

and

$$d_i^* = T_i \quad \text{for } i=k-1, \ldots, 1 \quad \text{for some } N \le k \le 1$$

where $\sum_{i=1}^{N} d_i^* = X_N$ and d_i^* is the optimal value of d_i.

Proof $C_1 p_1$ is the largest coefficient in the objective function and T_o can be increased most by making d_1 as large as possible since the coefficients of the decision variables in the first constraint are all equal to one. Likewise, $C_2 p_2$ is the next largest coefficient and resources should be allocated to d_2 whenever d_1 is at a constrained maximum. This reasoning can be continued up through $C_N p_N$ which completes the proof of Theorem 1.

In concept then, the solution is straightforward so long as the coefficients are in descending order. However, obtaining equations for the decision variables in order to allow direct computation is a more involved process.

c. Additional Notation

Let W_n represent the expected number of ABM's required to defend all undestroyed silos from stage n to stage one given there are T_n undestroyed silos at the beginning of stage n. T_n can have any value from zero to T_N.

Let Q_n represent the expected number of ABM's required to defend all undestroyed silos from stage n to stage one given <u>no</u> ABM's were launched prior to stage n.

d. Analysis

From the definitions above, it is obvious that

$$W_1 = T_1 \tag{3.19}$$

Suppose $X_N = W_1$. Then the defense can achieve 100% defense on stage one. Next, suppose $X_N > W_1$. In this case, those ABM's not required on stage one are allocated to stage two. However, as ABM's are allocated to stage two, the upper bound on d_1 increases (see constraint (2)) due to the increase in T_1.

Let T_{ij} represent the expected number of undestroyed silos immediately prior to stage i given defense started on stage j, j > i. Then for 100% defense on both stages one and two

$$W_2 = T_2 + T_{12} \tag{3.20}$$

But T_{12} is related to T_2 in the manner of eq. (2.5), specifically

$$T_{12} = T_2(1-q) + qp_2d_2. \tag{3.21}$$

Substituting eq. (3.21) into eq. (3.20) yields

$$W_2 = T_2[1 + (1-q)] + qp_2d_2.$$

However, from the definition of W_2 (100% defense), d_2 in this instance equals T_2. Thus

$$W_2 = T_2\ [1 + (1-q + qp_2)]. \tag{3.22}$$

In similar fashion, the expression for W_3 can be written

$$W_3 = T_3\ [1 + (1-q + qp_3) + (1-q + qp_3)\ (1-q + qp_2)]. \tag{3.23}$$

Lemma 3 For any stage j, $N \leq j \leq 1$,

$$W_j = T_j \left(1 + \sum_{i=1}^{j-1} \prod_{k=1}^{i} (1-q+qp_{j-k+1})\right). \qquad (3.24)$$

Proof For j=1, eq. (3.24) reduces to

$$W_1 = T_1$$

which is eq. (3.19). Thus Lemma 3 is valid for j=1.

It is assumed that Lemma 3 is valid for j=n. Consider stage n+1:

$$W_{n+1} = T_{n+1} + T_{n,n+1} + \ldots + T_{1,n+1} \qquad (3.25)$$

But $T_{n,n+1}$ is a particular value for T_n. Therefore, from Lemma 3 the number of ABM's required to defend 100% from stage n to stage one is

$$W_n = T_{n,n+1} \left(1 + \sum_{i=1}^{n-1} \prod_{k=1}^{i} (i-q + qp_{n-k+1})\right).$$

Also

$$T_{n,n+1} = T_{n+1} (1-q) + qp_{n+1} d_{n+1}. \qquad (3.26)$$

So

$$W_n = \left(T_{n+1}(1-q) + qp_{n+1}d_{n+1}\right)\left(1 + \sum_{i=1}^{n-1} \prod_{k=1}^{i} (1-q + qp_{n-k+1})\right). \qquad (3.27)$$

Again from the definition of W_{n+1}

$$T_{n+1} = d_{n+1}$$

and

$$W_n = T_{n+1} (1-q + qp_{n+1}) \left(1 + \sum_{i=1}^{n-1} \prod_{k=1}^{i} (1-q + qp_{n-k+1})\right). \qquad (3.28)$$

Returning to eq. (3.25)

$$W_{n+1} = T_{n+1} + W_n$$

$$W_{n+1} = T_{n+1}\left(1 + (1-q + qp_{n+1})\ (1 + \sum_{i=1}^{n-1} \prod_{k=1}^{i} (1-q + qp_{n-k+1}))\right)$$

$$W_{n+1} = T_{n+1}\left(1 + \sum_{i=1}^{n} \prod_{k=1}^{i} (1-q + qp_{n-k+2})\right). \tag{3.29}$$

which is equivalent to eq. (3.24) with j = n+1. This completes the proof of Lemma 3.

Using the definition of Q_j and Lemma 3 produces

$$Q_j = \left(\frac{W_j}{T_j}\right) T_N(1-q)^{N-j} \tag{3.30}$$

since $T_j = T_N(1-q)^{N-j}$ if no ABM's are launched on stages n+1, ..., N. Using the first constraint and Lemma 3 with j=n-1 yields

$$X_N = d_N + T_{n-1}\left(1 + \sum_{i=1}^{n-2} \prod_{k=1}^{i} (1-q + qp_{n-k})\right).$$

Let A be the coefficient of T_{n-1} above. Then

$$X_N = d_n + T_{n-1}A.$$

Also

$$T_{n-1} = T_n(1-q) + qp_n d_n \quad \text{and}$$

$$T_n = T_N(1-q)^{N-n}$$

since no defense occurred on stages n+1, ..., N. Thus

$$X_N = d_n + \left(T_N(1-q)^{N-n+1} + qp_n d_n\right) A.$$

Solving for d^*_n gives

$$d^*_n = \frac{X_N - AT_N(1-q)^{N-n+1}}{1 + Aqp_n}\ . \tag{3.32}$$

Continuing

$$d^*_{n-1} = T_{n-1} = T_N(1-q)^{N-n+1} + qp_n d^*_n. \tag{3.33}$$

Similarly

$$d^*_{n-2} = d^*_{n-1}\ (1-q + qp_{n-1}). \tag{3.34}$$

In general

$$d^*_{n-j} = d^*_{n-j+1}\ (1-q + qp_{n-j+1}) \qquad j=2,\ \ldots,\ n-1. \tag{3.35}$$

This form of the solution is most practical for actual computations since the equations for each d^*_j in terms of initial parameters become very large and unwieldly.

e. Solution Algorithm

In summary, the solution algorithm is

(1) Compute Q_N, Q_{N-1}, ..., Q_1 in that order using eq. (3.30) until that pair of Q's which bracket X_N is found. This yields the value of n.

(2) Compute d^*_n using eq. (3.32).

(3) Compute d^*_{n-1} using eq. (3.33).

(4) Compute d^*_{n-j}; j=2, ..., n-1; using eq. (3.35).

In a practical sense, it is only necessary to compute n and then d^*_n and defend all undestroyed silos on each succeeding stage. Based on expected values, this allocation will produce the maximum survivability of the silos.

f. The Optimal Value of the Objective Function

The objective function, eq. (3.12), was

$$\max T_o = T_N(1-q)^N + q \sum_{j=1}^{N} (1-q)^{j-1} p_j d_j.$$

Because the optimal solution indicates that no ABM's are launched prior to stage n, all decision variables with subscripts greater than n are zero. Thus, the optimal value for the objective function can be expressed as

$$T_o^* = T_N(1-q)^N + q \sum_{j=1}^{n} (1-q)^{j-1} p_j d_j^*. \tag{3.36}$$

However, from eq. (3.33)

$$d_{n-1}^* = T_N(1-q)^{N-n+1} + qp_n d_n^* \tag{3.37}$$

and from eq. (3.34)

$$d_{n-2}^* = d_{n-1}^*(1-q + qp_{n-1})$$

or

$$d_{n-2}^* = \left(T_N(1-q)^{N-n+1} + qp_n d_n^*\right) (1-q + qp_{n-1}).$$

Similarly

$$d_{n-3}^* = (1-q + qp_{n-2})\ (1-q + qp_{n-1}) \left(T_N(1-q)^{N-n+1} + qp_n d_n^*\right)$$

<u>Lemma 4</u> For every j; j=1, ..., n-1;

$$d_{n-j}^* = \prod_{h=1}^{j=1} (1-q + qp_{n-h}) \left(T_N(1-q)^{N-n+1} qp_n d_n^*\right). \tag{3.38}$$

<u>Proof</u> Lemma 4 will be proved by induction. For j=1, eq. (3.38) reduces to

$$d_{n-1}^* = T_N(1-q)^{N-n+1} + qp_n d_n^*$$

which is the same as eq. (3.37). Thus Lemma 4 is valid for j=1. Assume that Lemma 4 is valid for j=m. Consider $d_{n-(m+1)}^*$. From eq. (3.35)

$$d_{n-(m+1)}^* = d_{n-m}^*\ (1-q + qp_{n-m}).$$

But from Lemma 4

$$d_{n-m}^* = \prod_{h=1}^{m-1} (1-q + qp_{n-h}) \left(T_N(1-q)^{N-n+1} + qp_n d_n^*\right).$$

Thus

$$d^*_{n-(m+1)} = (1-q + qp_{n-m}) \prod_{h=1}^{m-1} (1-q + qp_{n-h}) \left(T_N(1-q)^{N-n+1} + qp_n d^*_n\right).$$

This can be written

$$d^*_{n-(m+1)} = \prod_{h=1}^{m} (1-q + qp_{n-h}) \left(T_N(1-q)^{N-n+1} + qp_n d^*_n\right)$$

which is the same form as Lemma 4 where j=m+1. This completes the proof of Lemma 4.

Returning to eq. (3.36) and substituting eq. (3.38) for each d^*_j; $j=1, \ldots, n-1$; yields

$$T^*_o = T_N(1-q)^N + q(1-q)^{n-1} p_n d^*_n + BC \tag{3.39}$$

where B and C are defined as follows

$$B = q \sum_{j=1}^{n-1} (1-q)^{j-1} p_j \left(\prod_{h=1}^{n-(j+1)} (1-q + qp_{n-h}) \right)$$

$$C = T_N(1-q)^{N-n+1} + qp_n d^*_n.$$

Using eq. (3.32) and substituting for d^*_n yields, after factoring

$$T^*_o = T_N(1-q)^{N-n+1}\left((1-q)^{n-1} + B\right) + \frac{X_N - AT_N(1-q)^{N-n+1}}{1 + Aqp_n} \; qp_n \left(B+(1-q)^{n-1}\right) . \tag{3.40}$$

Equation (3.40) is the value of T^*_o as expressed in terms of initial parameters and n. This is a cumbersome form for calculations, but it is developed here for use in a later section. See section 7 for a more direct means of calculating T^*_o.

6. <u>Discussion</u>

a. The Use of Expected Values

With the aid of computers, it would be possible and indeed worthwhile, to recompute an entirely new solution following each stage. This would result in a new d_n^* after each stage, and avoid the inherent discrepancies resulting from an analysis based on expected values.

It is unlikely that the actual number of undestroyed silos present at any stage of a real attack would be the same as the expected number. It must be remembered that the parameters q and N are estimates based on the latest intelligence information available and are subject to error. Even the p_i values, for which some data can be gathered, are subject to error. ABM's have never been launched or radars tested under conditions of severe atmospheric disturbance as would be caused by numerous nuclear blasts. Therefore, if an updated solution based on actual values could be obtained between stages, better results could be expected than by rigidly applying the complete initial solution.

Consider the situation in which an updated solution cannot be obtained between stages. Based on the foregoing analysis, the solution that should be used would be to defend on stage n with d_n^* ABM's. On each succeeding stage, defend all undestroyed silos, whether or not the number was greater or less than expected for that stage. If n were large, it is reasonable to assume that the number of stages requiring more ABM's than expected would have a cancelling effect on those stages requiring fewer ABM's than expected, and the shortage or excess of ABM's at the end of the attack would likely be small compared to X_N. However, if n were small the excess or shortage could be a sizeable proportion of X_N.

b. An Intuitively Appealing, but Erroneous Solution

When this problem was first considered, one apparent solution was to select a subset of T_N and defend that subset completely throughout the attack. Since it was assumed that a sufficient number of ABM's would not normally be available to defend all the silos properly, it was reasonable to assume that the number of ABM's available would be sufficient to defend a smaller portion of T_N. In general, of course, this is not true. However, if q equals one, this is precisely the solution one does obtain from model II. So for this limiting value of q, the intuitive solution is also correct. It would be foolish not to defend on stage N since every silo undefended in this case is destroyed. For q=1

$$Q_{N-1} = 0.$$

Therefore

$$Q_N \geq X_N \geq Q_{N-1}$$

for all permissable values of X_N. Thus, defense starts on stage N in all cases when q equals one.

c. The Estimate of N

The estimate of the parameter N is critical in this solution. It is necessary for the defense to determine N, and for a proper determination, the defense should be aware of the risks involved in overestimating and underestimating the parameter. If the estimated value for N turns out to be greater than the actual value, ABM's will be left over, thus wasting resources and losing more silos than necessary. On the other hand, if the estimated value for N is smaller than the actual value, the offense can attack unchallenged resulting in excessive losses of ICBM's.

Consider the case where q equals one. Underestimating N would be disasterous since one unchallenged stage results in complete loss of all remaining silos. Overestimating, on the other hand, would result in unused AMB's, but there would also be an expected number of undestroyed silos remaining. Conversely, for q close to zero, the opposite is true, everything else remaining unchanged. Other parameters also affect this balance. Without specific parameters to consider, it is difficult to make any quantitative statements about the erroneous estimation of N. Decision Theory and Game Theory both offer techniques for optimizing the choice of N. A great deal of additional study could be done on this facet of the problem.

7. A Numerical Example

Consider a four stage attack and let the initial parameters of the problem be as follows.

X_N = 340 ABM's

T_N = 150 silos (ICBM's)

$q = 0.8$

$p_4 = 0.9$

$p_3 = 0.75$

$p_2 = 0.65$

$p_1 = 0.6.$

a. Determining Q Values

From eq. (3.30)

$$Q_4 = (W_4/T_4)\ T_4(1-q)^{4-4}$$

or

$$Q_4 = W_4.$$

Substituting for W_4 using eq. (3.24)

$$Q_4 = T_4 \left(1 + \sum_{i=1}^{3} \prod_{k=1}^{i} (1 - 0.8 + 0.8p_{4-k+1})\right).$$

Using given parameter values

$$Q_4 = 150\ (1 + 0.92 + 0.74 + 0.53)$$

or $$Q_4 = 477.9.$$

Then

$$Q_4 \approx 478 > 340.$$

Similarly

$$Q_3 = (W_3/T_3)\ T_4(1-q)^{4-3}$$

$$Q_3 = 79.7.$$

So $$Q_3 \approx 80 < 340.$$

b. Determining n and d^*_n

Since $Q_4 > X_N > Q_3$, n=4, and from eq. (3.32)

$$d^*_n = \frac{340 - 150(0.2)\left(1 + \sum_{i=1}^{2} \prod_{k=1}^{i} (0.2 + 0.8p_{4-k})\right)}{1 + (0.8)\ (0.9)\left(1 + \sum_{i=1}^{2} \prod_{k=1}^{i} (0.2 + 0.8p_{4-k})\right)}$$

$$d^*_n = 99.1 \approx 99.$$

c. Completing the d's

From eq. (3.33)

$$d^*_3 = T_4(0.2) + (0.8)(0.9)(99)$$

$$d^*_3 = 101.3 \approx 101.$$

From eq. (3.35)

$$d^*_2 = d^*_3\ (1 - 0.8 + 0.8p_3)$$

$$d^*_2 = 101(0.2 + 0.6)$$

$$d^*_2 = 80.8 \approx 81.$$

And also from eq. (3.35)

$$d_1^* = 58.4 \approx 59.$$

Note that $\sum_{i=1}^{4} d_i^* = 340.$

d. Expected Number of Silos Remaining

Using eq. (3.12), the objective function, and the results above, the maximum expected value for T_o can be determined.

$$\max T_o = 150(0.2)^4 + 0.8 \sum_{j=1}^{4} (0.2)^{j-1} p_j d_j^*$$

$$\max T_o = 0.24 + 28.32 + 8.43 + 2.43 + 0.57$$

$$\max T_o = 39.98 \approx 40.$$

In summary, the solution above instructs the defense to launch 99 ABM's on stage four, 101 on stage three, 81 on stage two, and 59 on the final stage. This allocation will result in saving approximately 40 silos from destruction.

As a sidelight, notice that the expected number of silos saved from destruction may be computed more directly without the use of the original objective function. Because of the nature of the solution, the following method is valid. Since $d_1^* = 59$ ABM's, there are expected to be 59 undestroyed silos at the beginning of the last stage. Also, $p_1 = 0.6$, so 35.4 or approximately 36 RV's are destroyed on stage one. Thus 23 RV's are expected to successfully pass through the defense, but of these only 80% are expected to be successful in destroying their targets. Therefore 35.4 plus 4.6 silos or 40 silos, are not destroyed on stage one. This is the same result one would obtain if eq. (3.35) were extended to include j=n, with $d_o^* = T_o^*$. d_o^*

has no physical meaning, but the pattern of the solution allows for this simple extension. Thus

$$T_o^* = d_1^*(1 - q + qp_1)$$

which algebraically carries out the same operations which were performed verbally above.

8. Maximum Number of ABM's Allowable

Recall that the models presented in this paper are all concerned with single ABM launchings. Therefore, there exists an upper bound on the number of ABM's which can be used in any anticipated attack. Should the defense have a greater number of ABM's than this upper bound, multiple launchings would very likely be preferred over single launchings and the solution previously presented cannot be rigidly applied. It remains to determine this upper bound.

a. Development

The maximum number of ABM's the defense can launch, using single launchings only, is that number for which every stage is defended completely, i.e. every RV aimed at an undestroyed silo is challenged by one ABM on every stage of the attack. Thus

$$d_N^* = T_N$$

and

$$(X_N)_{max.} = W_N$$

since 100% defense occurs on each stage following stage N because of the nature of the optimal solution. From Lemma 3

$$(X_N)_{max.} = T_N \left(1 + \sum_{i=1}^{N-1} \prod_{k=1}^{i} (1-q + qp_{N-k+1})\right).$$

Thus any number of ABM's equal to or less than $(X_N)_{max.}$ is acceptable. If the ratio X_N/T_N satisfies the following inequality, the solution given in this paper can be used.

$$X_N/T_N \leq 1 + \sum_{i=1}^{N-1} \prod_{k=1}^{i} (1-q + qp_{N-k+1}).$$

For the example of section 7

$$X_N/T_N = 340/150 = 2.3$$

and

$$1 + \sum_{i=1}^{3} \prod_{k=1}^{i} (0.2 + 0.8p_{5-k}) = 1 + 0.92 + 0.74 + 0.53$$

$$'' \quad '' \quad '' \quad '' \quad = 3.2.$$

Since 2.3 < 3.2, the solution presented in this paper is applicable.

9. Single Launchings Versus Multiple Launchings

One of the assumptions made in this study was that only one ABM would be launched in defense of an undestroyed silo on any particular stage, and that multiple launchings of ABM's would never be preferred over single launchings. The term "multiple launchings" used throughout this paper means the use of more than one ABM against a single RV aimed at an undestroyed silo on a given stage. The validity of this assumption is dependent upon the values of the parameters of the problem. The purpose of this section is to devise a method for determining when the assumption is valid. Multiple launchings of three or more ABM's per silo per stage will not be considered, since this study is only concerned with those cases where single launchings are optimal. It follows that if single launchings are preferred over double launchings, then they are preferred over any other type of multiple launch.

Consider eq. (3.40). An increase in X_N causes an increase in T_o^* given by

$$\Delta T_o^* = \frac{qp_n(B + (1-q)^{n-1})}{1 + Aqp_n} \Delta X_N. \qquad (3.41)$$

A question one might ask is, given one additional ABM, should it be used according to the solution algorithm presented in this paper or should it be used to increase the defense of an undestroyed silo at stage one? To answer this question it is necessary to compare the marginal gain in T_o^* resulting from both options.

If $\Delta X_N = 1$, then ΔT_o^* is equal to the coefficient of ΔX_N in eq. (3.41) where ΔT_o^* is the increase in the objective function if a new optimal solution (for single launchings) is computed. In the case where the additional ABM is used on stage one, the increase in T_o^* (call it D) is equal to the probability that a particular undestroyed silo survives stage one given that two ABM's are launched in its defense minus the probability that the same silo survives stage one given only one ABM is launched in its defense. Thus

$$\begin{aligned} D &= \left(1 - q\ (1-p_1)^2\right) - \left(1 - q\ (1-p_1)\right) \\ D &= qp_1(1-p_1). \end{aligned} \qquad (3.42)$$

Consider the case where n=1, that is, where defensive firing begins on stage one. Using eq. (3.41)

$$\Delta T_o^* = \frac{qp_1^{\,2} + qp_1}{1 + qp_1} = \frac{qp_1(1 + q)}{1 + qp_1}.$$

Also $\qquad D = qp_1(1 - p_1).$

Note that $\Delta T_o^* \geq D$

since $(1 + q)/(1 + qp_1) \geq 1$ and $(1 - p_1) \leq 1$, which is to say that for $p_1 > 0$, single launchings are always preferred over double launchings on stage one given that defense begins on stage one.

Next consider the case where n=2. Then

$$\Delta T_o^* = \frac{q^2 p_2 p_1 + q(1-q)p_2}{1 + qp_2} = \frac{qp_2(1 - q + qp_1)}{1 + qp_2}.$$

D does not change since the same change in X_N is being considered throughout. Thus

$$D = qp_1(1-p_1).$$

The assumption of single launchings is valid in this case so long as

$$\frac{p_2(1 - q + qp_1)}{1 + qp_2} \geq p_1(1-p_1).$$

Therefore, for any specific set of parameters the following general comparison may be made to determine if the set of parameter values are such as to justify the use of the solution presented in this paper. If

$$\Delta T_o^* \geq qp_1(1-p_1)$$

for that value of n as determined by step one of the solution algorithm, then the initial assumption denying multiple launchings is valid on probabilistic grounds. There may be physical, engineering, or other restrictions which prohibit multiple launchings.

Using the parameter values as given for the example of section 7, where n=4

$$\Delta T_o^* = 0.104 \qquad \text{and} \qquad D = 0.192$$

which indicates that at least one of the ABM's designated for stage four should be used on stage one if there is no valid reason to prohibit multiple launchings. Thus, the example, although sufficient for displaying the use of the solution algortihm, does not produce a maximum survivability if multiple launchings are allowed. Therefore, from a probabilistic point of view, there are cases where multiple launchings are desirable even though some silos are undefended in the early stages of the attack.

IV. EXTENSIONS OF THE PROBLEM

A. MULTIVALUED TARGETS

Since this section involves equations which are analogous to model II, and because a detailed analysis of this subject is not intended, only those equations which characterize the problem and those comments which could be useful for obtaining a solution are given. The next section assumes silos of three different values for discussion purposes but any number of values (up to T_N) is possible.

Consider each silo of the farm to have associated with it a value, v_I, v_{II}, or v_{III} where, without loss of generality, it can be assumed that

$$v_I < v_{II} < v_{III}.$$

Such values can be related to the retalitory capabilities of the ICBM's located in the silos.

There are now three distinct classes of silos to be considered, with T_N^I, T_N^{II}, and T_N^{III} numbers of silos in each class. The total number of silos in the farm is just the sum of the numbers of silos in each class,

$$T_N = T_N^I + T_N^{II} + T_N^{III}.$$

All other assumptions of the basic problem are valid in this generalized case. Let $(TW)_{N-i}$ be defined as the total worth of all undestroyed silos immediately prior to stage (N-i). Then

$$(TW)_N = v_I T_N^I + v_{II} T_N^{II} + v_{III} T_N^{III}$$

1. The Objective Function

The purpose of the ABM system is to defend the farm so as to maximize the capability for retaliation. Thus, in a development analogous to the development of eq. (3.12), the generalized objective function is

$$\max\ (TW)_o = \sum_{m=I}^{III} v_m \left[(1-q)^N T_N^m + q \sum_{j=1}^{N} (1-q)^{j-1} p_j d_j^m \right] \qquad (4.1)$$

where d_j^I, d_j^{II}, and d_j^{III} are the decision variables representing the numbers of ABM's to be launched in defense of each of the three classes of silos on stage j. Such an objective function maximizes the total capability for retaliation following an attack.

2. The Constraints

The generalized constraints can be written directly from eq. (3.16) of model II. They are

$$(1) \quad \sum_{m=I}^{III} \sum_{i=0}^{N-1} d_{N-i}^m = X_N$$

$$(2) \quad d_{N-i}^m - q \sum_{j=N-i+1}^{N} (1-q)^{j-(N-i+1)} p_j d_j^m \leq T_N^m (1-q)^i \qquad (4.2)$$

$$i=1, \ldots, N-1$$

$$m=I, III, III$$

$$(3) \quad d_{N-i}^m \geq 0 \qquad i=1, \ldots, N-1$$

$$m=I, II, III$$

3. Analysis

Note that, in eq. (4.1), the quantity

$$(1-q)^N \sum_{m=I}^{III} v_m T_N^m$$

is fixed for a given set of parameters. Thus, optimization (maximization of $(TW)_o$) shall be concerned only with the remaining terms. Let qC_j^m be the coefficient of d_j^m in each of the remaining terms, i.e.

$$C_j^m = v_m(1-q)^{j-i}p_j.$$

As in model II, an evaluation of the C_j^m factors leads to the optimal solution provided

$$p_i/p_{i+1} > (1-q) \text{ (see section III.B.5).}$$

One useful way to compare the C_j^m's is in the form of a matrix where $a_{jm} = C_j^m$. Each row of the matrix relates to a stage of the attack and each column is associated with a distinct class of silos. Thus, the matrix for this section is N x 3 whereas the matrix for model II is an N-dimensional vector.

Assume that each element of the matrix is different. Then there exists an ordered sequence of matrix elements such that the first member of the sequence is the largest element of the matrix, the second member of the sequence is the second largest element of the matrix, and so on. With the earlier assumption that $v_I < v_{II} < v_{III}$, the first ABM's would be assigned to defend silos of class III on stage one. The analysis could continue as in model II, but a separate algorithm would be necessary for every feasible sequence of elements. The number of feasible sequences increases rapidly as the number of classes of silos increases and the method presented in this paper becomes inefficient. A more general approach is necessary to obtain an efficient solution to this generalized problem. Obtaining such a solution is recommended as a topic for further study.

B. OTHER TARGETS

It was assumed throughout this paper that the ABM's and associated defensive equipment such as the radars and computers were immune to attack by enemy RV's. Of course, in a real situation this probably would not be true. The enemy might decide it would be to his advantage to strike the defensive system first and then use fewer RV's on the silos. In such a case a much larger problem exists than was presented earlier. It would <u>not</u> be sufficient to merely assign radars, computers, and ABM's values as targets since the values of radars, computers, and ABM's are all interdependent. If no ABM's are available, radars and computers are useless, and if the radars or computers are destroyed, unlaunched ABM's are also of no value. Also, a simple, saturation, offensive strategy could <u>not</u> be assumed for in this case there exists a trade-off between attacking silos and attacking the ABM system in terms of total silo destruction. It is only natural to assume that the enemy would analyze this trade-off and attack in such a manner as to maximize total silo destruction. Thus, for useful results, it would be necessary to optimize offensive <u>and</u> defensive strategies, simultaneously.

There are numerous other variations to the basic problem investigated by this paper. There are also numerous other methodologies available (game theory, decision theory, computer simulation) for formulating, analyzing, and solving the problem variations. The ABM question is a fertile field for analysis and is sure to receive a great amount of attention in the future. This paper has been a modest attempt at solving a simple form of a complex problem, and has tried to present some insight into the nature of the greater problem.

APPENDIX A

THE ANALYTICAL DIFFICULTY OF FRACTIONAL ABM'S

Let P represent the probability that a target survives three consecutive, independent, attacking RV's given that a single ABM is launched in defense. Then

$$P = (1-q)\ (1-q) \left[1 - q\ (1-p)\right]$$

where q and p are defined as in part B, chapter II.

Let R represent the probability that a target survives three consecutive, independent, attacking RV's given that 1/3 of an ABM is launched at each RV. Then

$$R = \left[1 - q\ (1-p/3)\right]^3.$$

Now the problem remains to determine whether $P > R$ or $R \geq P$. Assume $R \geq P$. Then

$$\left[1 - q\ (1-p/3)\right]^3 \geq (1-q)^2 \left[1 - q\ (1-p)\right].$$

Expanding both sides and collecting terms yields

$$\text{LHS} = 1 - 3q + 3q^2 - q^3 + pq + pq^3 - 2pq^2 + p^2q^2/3 - p^2q^3/3 + p^3q^3/27.$$

$$\text{RHS} = 1 - 3q + 3q^2 - q^3 + pq + pq^3 - 2pq^2.$$

The first seven terms of the LHS cancel the seven terms of the RHS, leaving

$$p^2q^2/3 - p^2q^3/3 + p^3q^3/27 \geq 0.$$

But $\quad p^2q^2/3 \geq p^2q^3/3$

since $\quad 0 \leq q \leq 1$

and $\quad p^3q^3/27 \geq 0.$

Thus $R = P$ only if $p = 0$ and $q = 1$. Such values taken together are totally outside the realm of interest. Thus it can be concluded, for $p > 0$, and any $0 \leq q \leq 1$,

$$R > P.$$

So although it is mathematically convenient to do calculations using fractional ABM's, it may well lead to inflated values for target survivability.

APPENDIX B

DETAILS IN THE DEVELOPMENT OF MODEL I

A. REDEFINING THE OBJECTIVE FUNCTION

From eq. (3.1)

$$z = \sum_{j=1}^{N} Y_j \tag{B.1}$$

where $Y_j = qT_j - qp_jd_j$ (eq. (2.4)).

Therefore

$$\begin{aligned} Y_1 &= qT_1 - qp_1d_1 \\ Y_2 &= qT_2 - qp_2d_2 \\ &\vdots \\ Y_N &= qT_N - qp_Nd_N \end{aligned} \tag{B.2}$$

and

$$z = q\sum_{j=1}^{N} (T_j - p_jd_j). \tag{B.3}$$

However, from Lemma 1

$$T_j = T_N(1-q)^{N-j} + q\sum_{k=j+1}^{N} (1-q)^{k-(j+1)} p_kd_k. \tag{B.4}$$

So (B.3) becomes

$$z = q\sum_{j=1}^{N} \left[T_N(1-q)^{N-j} + q\sum_{k=j+1}^{N} (1-q)^{k-(j+1)} p_kd_k\right] - q\sum_{j=1}^{N} p_jd_j$$

$$z = qT_N \sum_{j=1}^{N} (1-q)^{N-j} + q^2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} (1-q)^{k-(j+1)} p_kd_k - q\sum_{j=1}^{N} p_jd_j$$

$$z = qT_N \sum_{j=1}^{N} (1-q)^{N-j} + q\sum_{j=1}^{N} \left[q\sum_{k=j+1}^{N} (1-q)^{k-(j+1)} p_kd_k - p_jd_j\right] \tag{B.5}$$

Letting N-j=i, the first term can be written

$$qT_N \sum_{i=0}^{N-1} (1-q)^i$$

which is identical to the expression for D_o in eq. (3.5). It now remains to separate the decision variables. Note that d_1 is in only one term of the expansion, d_2 is in two terms, d_3 is in three terms, and so on.

Expanding the bracketed part of eq. (B.5) yields, for

$$\begin{array}{ll}
j=1: & qp_2d_2 + q(1-q)^1 p_3d_3 + q(1-q)^2 p_4d_4 + \ldots + q(1-q)^{N-2} p_Nd_N - p_1d_1 \\
j=2: & q(1-q)^0 p_3d_3 + q(1-q)^1 p_4d_4 + \ldots + q(1-q)^{N-3} p_Nd_N - p_2d_2 \\
j=3: & q(1-q)^0 p_4d_4 + \ldots + q(1-q)^{N-4} p_Nd_N - p_3d_3 \\
\vdots & \vdots \\
j=N-1: & q(1-q)^0 p_Nd_N - p_{N-1}d_{N-1} \\
j=N: & - p_Nd_N
\end{array}$$

Summing terms common to each d_j and multiplying through by q produces

$$-qp_1d_1 - qp_2(1-q)d_2 + qp_3\left[q\sum_{t=0}^{1}(1-q)^t - 1\right]d_3 + \ldots + qp_N\left[q\sum_{t=0}^{N-2}(1-q)^t - 1\right]d_N. \tag{B.6}$$

The first term above is D_1 as used in eq. (3.5). The general expression for any of the terms above involving $d_2, \ldots, d_N$ can be written

$$qp_k\left[q\sum_{t=0}^{k-2}(1-q)^t - 1\right]d_k$$

which is equivalent to the general term in eq. (3.5) when the substitutions t=j and k=N-i are used. The proof of eq. (B.6) will not be given. It is analogous to the inductive proofs used for Lemmas 1 and 3.

In conclusion, the objective function can be written

$$z = \sum_{j=1}^{N} Y_j \text{ or equivalently,}$$

$$z = D_o + D_1 d_1 + D_2 d_2 + \ldots + D_N d_N$$

with the coefficients of the d_j's as defined above.

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

The Optimum Use of Antiballistic Missiles for Point Target Defense

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Master's Thesis; October 1969

5. AUTHOR(S) (First name, middle initial, last name)

LeRoy George Krumm

6. REPORT DATE October 1969	7a. TOTAL NO. OF PAGES 51	7b. NO. OF REFS 6
8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

This thesis is concerned with developing an optimal launching schedule for ABM's deployed for defense of a number of point targets, i.e. an ICBM complex. Let the attack occur in N stages with an offensive strategy of saturation. A dynamic programming model is developed for formulating the problem and a linear programming model is used in its solution. Equations are developed for determining the optimal number of ABM's to launch on each stage of the attack so as to maximize the expected number of silos surviving. Multivalued point target defense is discussed and formulated but no specific solutions are offered. The ABM system (including radars, computers, etc.) is not considered subject to attack. Some discussion of this aspect of the problem is offered. Single ABM launchings per re-entry vehicle are assumed. A test is developed to determine for what parameters single launchings are preferred over multiple launchings, under the assumptions of the attack.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Security Classification

A-31408

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Antiballistic missile allocation						
Point target defense						
Missiles						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

Security Classification

A-31409